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Widom Factors

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Abstract Given a non-polar compact set K, we define the n-th Widom factor $W_n(K)$ as the ratio of the sup-norm of the n-th Chebyshev polynomial on K to the n-th degree of its logarithmic capacity. By G. Szegő, the sequence $(W_n(K))_{n=1}^{\infty}$ has subexponential growth. Our aim is to consider compact sets with maximal growth of the Widom factors. We show that for each sequence $(M_n)_{n=1}^{\infty}$ of subexponential growth there is a Cantor-type set whose Widom's factors exceed M_n. We also present a set K with highly irregular behavior of the Widom factors.

Keywords Logarithmic capacity · Chebyshev numbers · Cantor sets

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1 Introduction

Let K be a compact subset of $\mathbb{C}$ consisting of infinitely many points. By $T_{n,K}$ we denote the corresponding Chebyshev polynomial, that is the unique monic polynomial of degree n for which its supremum norm $t_n(K) := ||T_{n,K}||_K$ is minimal among all monic polynomials of the same degree. By M. Fekete [4], there exists $\lim_{n\to\infty} t_n(K)^{\frac{1}{n}}$.
G. Szegő [9] showed that this limit coincides with $Cap(K)$,the logarithmic capacity of K. It is a consequence of logarithmic subadditivity of Chebyshev's numbers $t_n(K)$ that

$$t_n(K) \geq Cap^n(K).$$

This inequality is sharp, since $t_n(\overline{\mathbb{D}}) = 1$ and $Cap(\overline{\mathbb{D}}) = 1$.

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Let us define *n-th Widom factor* of a non-polar compact set $K \subset \mathbb{C}$ as

$$W_n(K) := \frac{t_n(K)}{Cap^n(K)}.$$

Thus, $W_n(K) \geq 1$ and the sequence $(W_n(K))_{n=1}^{\infty}$ has subexponential growth, that is $n^{-1} \ln W_n(K) \to 0$ as $n \to \infty$. The problem of behavior of Widom's factors attracted the attention of many researches. Section 2 is a brief review of relevant results.

Usually the values $W(K) := \liminf_n W_n(K)$ are estimated for different compact sets. Here we analyze the case of maximal growth of the Widom factors. In Section 3 we calculate $W_{2^s}(K(\gamma))$ for *weakly equilibrium Cantor-type sets* introduced in [5].

In Section 4 we show that for each sequence $(M_n)_{n=1}^{\infty}$ of subexponential growth there is $K(\gamma)$ whose Widom's factors exceed M_n. Thus, it is not possible to find a sequence $(M_n)_{n=1}^{\infty}$ of subexponential growth and a constant $C>0$ such that the inequality

$$W_n(K) \leq C \cdot M_n$$

is valid for all non-polar compact sets and for all $n \in \mathbb{N}$.

In the last section we construct a Cantor-type set K with highly irregular behavior of Widom's factors. Namely, one subsequence of $(W_n(K))_{n=1}^{\infty}$ converges (as fast as we wish) to the value 2, which is the smallest possible accumulation point for $W_n(K)$ if $K \subset \mathbb{R}$, whereas another subsequence exceeds any sequence $(M_n)_{n=1}^{\infty}$ of subexponential growth given beforehand.

For basic notions of logarithmic potential theory we refer the reader to [7], log denotes the natural logarithm.

2 Some Estimations of Widom's Factors

Here we give a brief exposition of the relevant material in our terms. Exact values of $W_n(K)$ for all n are known only for a few cases. For instance, $W_n(\overline{\mathbb{D}}) = 1$ and $W_n([-1, 1]) = 2$ for all $n \in \mathbb{N}$.

An easy computation shows that Widom's factors are invariant under dilation and translation:

$$W_n(\lambda K + z) = \frac{t_n(\lambda K + z)}{Cap^n(\lambda K + z)} = \frac{\lambda^n \ t_n(K)}{(\lambda \ Cap(K))^n} = W_n(K),$$

where $\lambda > 0,\ z \in \mathbb{C}$.

N.I. Achieser showed in [1] and [2] that, even in simple cases, the behavior of the sequence $(W_n(K))_{n=1}^{\infty}$ is rather irregular.

Theorem 2.1 ([1, 2]) *Let K be a union of two disjoint closed intervals. If there exists a polynomial P_n such that $P_n^{-1}([-1, 1]) = K$, then $(W_n(K))_{n=1}^{\infty}$ has a finite number of accumulation points from which the smallest is 2.*

Otherwise, if there is no P_n with $P_n^{-1}([-1, 1]) = K$, then the accumulation points of $(W_n(K))_{n=1}^{\infty}$ fill out an entire interval of which the left endpoint is 2.

Thus, $W([a, b] \cup [c, d]) = 2$. In 2008 K. Schiefermayr generalized Theorem 2.1 to any real compact set.

Theorem 2.2 *([8], T.2) Let $K \subset \mathbb{R}$ be a non-polar compact set. Then $W_n(K) \geq 2$ for each $n \in \mathbb{N}$, where $W(K) = 2$ if K is a polynomial preimage of* $[-1, 1]$.

Recently, V. Totik showed that the interval is the only real compact set for which $W_n(K)$ converge to 2.

Theorem 2.3 ([12], *T.3*) *If* $K \subset \mathbb{R}$ *is not an interval, then there is a* $c>0$ *and a subsequence* $\mathcal{N}$ *of the natural numbers such that* $W_n(K) \geq (2+c)$ *for* $n \in \mathcal{N}$.

Specifically, in the case when K is a finite union of disjoint intervals, V. Totik found the best possible rate of convergence of subsequences from $(W_n(K))_{n=1}^{\infty}$.

Theorem 2.4 ([11], *T.3*) *Let* $K \subset \mathbb{R}$ *be a compact set consisting of* l *intervals. Then there is a constant* C *such that for infinitely many* n

$$W_n(K) \leq 2(1 + C\, n^{-1/(l-1)}).$$

This upper bound is the best possible, because of the following result.

Theorem 2.5 ([11], *T.4*) *For every* $l > 1$ *there are a set* K *consisting of* l *intervals and a constant* $c > 0$ *such that for all* n

$$W_n(K) > 2(1 + c\, n^{-1/(l-1)}).$$

We suggest the name *Widom factor* for $W_n(K)$ because of the fundamental paper [13], where H. Widom considered $K \subset \mathbb{C}$ that are finite unions of smooth Jordan curves and showed that $W(K) = 1$ for this case. Earlier, G. Faber in [3], using polynomials now named after him, showed that $W_n(K) \to 1$ as $n \to \infty$ for a single analytic curve K. For a treatment of more general complex compact sets we refer the reader to [12].

Concerning the upper bounds for $(W_n(K))_{n=1}^{\infty}$, H. Widom showed in [13] that in the case of finite unions of disjoint intervals, this sequence is bounded. Actually, he did not explicitly present this result, but V. Totik stated it as a consequence of Theorem 11.5 in [13] and gave another proof using polynomial inverse images.

Theorem 2.6 ([10], *T.1*) *Let* $K \subset \mathbb{R}$ *consist of finitely many disjoint intervals. Then there is a constant* C *depending only on* K *such that* $W_n(K) \leq C$ *for all* $n \in \mathbb{N}$.

Let us show that for some Cantor-type sets the sequence $(W_n(K))_{n=1}^{\infty}$ is unbounded and any subexponential growth can be achieved.

3 Widom's Factors for Weakly Equilibrium Cantor-type Sets

For the convenience of the reader we repeat the relevant material from [5]. Given sequence $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s < 1/4$, let $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$ for $s \in \mathbb{N}$. Define $P_2(x) = x(x-1)$ and $P_{2^{s+1}} = P_{2^s}(P_{2^s} + r_s)$ for $s \in \mathbb{N}$. Then, by Lemma 1 in [5], P_{2^s}' has $2^s - 1$ simple zeros, where 2^{s-1} of them are minima of P_{2^s} with equal values $P_{2^s} = -r_{s-1}^2/4$ and remaining $2^{s-1} - 1$ extrema are local maxima of P_{2^s} with positive values.

Consider the set $E_s := \{x \in \mathbb{R} : P_{2^{s+1}}(x) \leq 0\} = \cup_{j=1}^{2^s} I_{j,s}$. The s-th level intervals $I_{j,s}$ are disjoint and $\max_{1 \leq j \leq 2^s} |I_{j,s}| \to 0$ as $s \to \infty$. Since $E_{s+1} \subset E_s$, we have a Cantor type set $K(\gamma) := \cap_{s=0}^{\infty} E_s$. In favor of this set, in comparison to usual Cantor-type sets, $K(\gamma)$

represents an intersection of polynomial inverse images of intervals. Indeed, the set E_s can also be presented as $(\frac{2}{r_s} P_{2^s} + 1)^{-1}([-1, 1])$. For many interesting properties of polynomial inverse images and for a characterization of polynomial inverse images of intervals see e.g. [6] and other related papers by F. Peherstorfer.

The set $K(\gamma)$ is non-polar if and only if

$$\sum_{n=1}^{\infty} 2^{-n} \log \frac{1}{\gamma_n} < \infty, \tag{3.1}$$

where the last sum gives the value of the Robin constant for the set $K(\gamma)$. By Proposition 1 in [5], the polynomial $P_{2^s} + r_s/2$ is the 2^s−th degree Chebyshev polynomial on $K(\gamma)$. From here we get

Proposition 3.1 *Assume* $(\gamma_s)_{s=1}^{\infty}$ *with* $0 < \gamma_s < 1/4$ *satisfies* (3.1). *Then for* $s \in \mathbb{Z}_+$

$$W_{2^s}(K(\gamma)) = \frac{1}{2} \exp\left(2^s \sum_{n=s+1}^{\infty} 2^{-n} \log \frac{1}{\gamma_n}\right). \tag{3.2}$$

Indeed, $Cap(K(\gamma)) = \exp(-Rob(K(\gamma)) = \exp\left(\sum_{n=1}^{\infty} 2^{-n} \log \gamma_n\right)$. On the other hand, $t_{2^s}(K(\gamma)) = ||P_{2^s} + r_s/2||_{K(\gamma)} = r_s/2 = \frac{1}{2} \exp\left(2^s \sum_{n=1}^{s} 2^{-n} \log \gamma_n\right)$,as is easy to check.

Now we can present a compact set with unbounded sequence of Widom's factors.

Example 3.2 For a fixed $M > 4$, let $\gamma_s = M^{-s}$ for $s \in \mathbb{N}$. Then $W_{2^s}(K(\gamma)) = M^{s+2}/2$.

By Theorem 3 in [5], in the case $\inf \gamma_s > 0$,the set $K(\gamma)$ is uniformly perfect. Recall that a compact set K is uniformly perfect if it has at least two points and the moduli of annuli in the complement of K which separate K are bounded.

Example 3.3 Assume $\gamma_0 \le \gamma_s < 1/4$ for $s \in \mathbb{N}$. Then $2 < W_{2^s}(K(\gamma)) \le 1/2\gamma_0$.

It is interesting that Proposition 3.1 and Example 3.3 are also valid in the limit case, when $\gamma_s = 1/4$ for some s. For example, let $\gamma_s = 1/4$ for all s (compare this with Example 1 in [5]). Then all local maxima of P_{2^s} are equal to 0. Therefore, $E_s = [0, 1]$ for each s, $K(\gamma) = [0, 1]$ and $W_n(K(\gamma)) = 2$ for all n. On the other hand, $T_{2^s, K(\gamma)}(x) = P_{2^s}(x) + r_s/2 = 2^{1-2^{s+1}} T_{2^s}(2x - 1)$ for $s \in \mathbb{N}$,where T_n stands for the classical Chebyshev polynomial, that is $T_n(t) = \cos(n \arccos t)$ for $|t| \le 1$. Thus, in the limit case, $t_{2^s}(K(\gamma)) = 2^{1-2^{s+1}}$. Since $Cap[0, 1] = 1/4$,we get $W_{2^s}(K(\gamma)) = 2$,which coincides with the value of the expression on the right in Eq. 3.2.

By Theorem 2.2, $W_n(K) \ge 2$ for any compact set on the line. Let us show that, for large Cantor sets $K(\gamma)$, the value 2 can be achieved by $W_{2^s}(K(\gamma))$ as fast as we wish (compare this with Theorems 2.4 and 2.5).

Theorem 3.4 *For each monotone null sequence* $(\sigma_s)_{s=0}^{\infty}$ *there is a Cantor set* K *such that* $W_{2^s}(K) = 2(1 + \sigma_s)$ *for all* s.

Proof Let us take $\gamma_n = 1/4 \cdot (1+\delta_n)^{-1}$, where $(\delta_n)_{n=1}^{\infty}$ will be defined later. Then

$$W_{2^s}(K(\gamma)) = \frac{1}{2}\exp\left[2^s \sum_{n=s+1}^{\infty} 2^{-n}(\log 4 + \log(1+\delta_n))\right] = 2\,\exp\left[\sum_{n=s+1}^{\infty} 2^{s-n}\log(1+\delta_n)\right].$$

This takes the desired value, if the system of equations

$$\sum_{n=s+1}^{\infty} 2^{s-n}\log(1+\delta_n) = \log(1+\sigma_s),\, s \in \mathbb{Z}_+$$

with unknowns $(\delta_n)_{n=1}^{\infty}$ is solvable. Multiplying the $s-$th equation by 2 and subtracting the $(s+1)-$th equation yields $\delta_{s+1} = (2\,\sigma_s + \sigma_s^2 - \sigma_{s+1})(1+\sigma_{s+1})^{-1}$ for $s \in \mathbb{Z}_+$. Since these values are positive, the set $K(\gamma)$ is well-defined. □

4 Widom's Factors of Fast Growth

First let us show how to construct $K \subset \mathbb{R}$ with preassigned values of a subsequence of the Widom factors. Recall that a sequence $(M_n)_{n=1}^{\infty}$ with $M_n \geq 1$ for $n \in \mathbb{N}$ has a subexponential growth if $\lim_{n\to\infty} \log M_n/n = 0$.

Proposition 4.1 *Suppose we are given a sequence $(M_n)_{n=1}^{\infty}$ of subexponential growth with $M_n > 1$ for all $n \in \mathbb{N}$ and a strictly monotone sequence $(\log M_n/n)_{n=1}^{\infty}$. Then there exists $K(\gamma)$ such that $W_{2^s}(K(\gamma)) = 2 \cdot M_{2^s}$ for $s \in \mathbb{Z}_+$.*

Proof Let us define $\beta_n = \log M_n/n$. Then $\beta_n \searrow 0$ and the series $\sum_{n=s+1}^{\infty}(\beta_{2^{n-1}} - \beta_{2^n})$ converges to β_{2^s}. By assumption, $M_{2^s} < M_{2^{s-1}}^2$ for all $s \in \mathbb{N}$. Let us take $\gamma_s = 4^{-1}\exp\left[-2^s(\beta_{2^{s-1}} - \beta_{2^s})\right] = 4^{-1}M_{2^s}/M_{2^{s-1}}^2$. Then $\gamma_s < 1/4$ for all $s \in \mathbb{N}$ and the set $K(\gamma)$ is well-defined and is not polar. By Proposition 3.1,

$$W_{2^s}(K(\gamma)) = \frac{1}{2}\exp\left[2^s \sum_{n=s+1}^{\infty} 2^{-n}(\ln 4 + 2^n(\beta_{2^{n-1}} - \beta_{2^n}))\right] = 2\exp(2^s\,\beta_{2^s}) = 2\,M_{2^s}.$$

□

Corollary 4.2 *For every C with $2 \leq C < \infty$ there exists $K \subset \mathbb{R}$ and a subsequence $\mathcal{N} \subset \mathbb{N}$ such that $W_n(K) = C$ for all $n \in \mathcal{N}$.*

Proof If $C = 2$ then K can be taken as any interval. If $C > 2$ then define $M_n = C/2$ for all n and apply the theorem. □

In the next theorem we show that any subexponential growth of Widom's factors can be exceeded for small sets $K(\gamma)$. We begin with the following regularization lemma.

Lemma 4.3 *For every sequence $(m_n)_{n=1}^{\infty}$ of subexponential growth and $\alpha < 1$ there is an increasing sequence $(M_n)_{n=1}^{\infty}$ of subexponential growth with $M_n = \exp(n \cdot \beta_n) \geq m_n$ for all n such that $\beta_1 \geq 1$, the sequence $(\beta_{2^s})_{s=0}^{\infty}$ decreases and $\beta_{2^s} \geq \alpha\beta_{2^{s-1}}$ for $s \in \mathbb{N}$.*

Proof First we take the increasing majorant $M_n = \sup_{k \leq n} m_k$. Then $M_n \nearrow$ and has subexponential growth. Since for each constant C the sequence $(M_n + C)_{n=1}^{\infty}$ is of subexponential growth, we can assume at once that $\beta_1 = \log M_1 \geq 1$. We can also suppose that $\beta_n \searrow 0$, since otherwise the replacement of β_n by $\sup_{k \geq n} \beta_k$ only enlarges $(M_n)_{n=1}^{\infty}$ and preserves its monotonicity. Thus it remains to provide the condition $\beta_{2^s} \geq \alpha\, \beta_{2^{s-1}}$ for $s \in \mathbb{N}$ and $\alpha < 1$. Without loss of generality we can assume $1/2 < \alpha$.

Fix the first $S \geq 0$ for which $\beta_{2^{S+1}} < \alpha\, \beta_{2^S}$. Let us take the new values $\tilde{\beta}_{2^{S+1}} = \alpha\, \beta_{2^S}$ and $\tilde{\beta}_{2^{S+k}} = \max\{\beta_{2^{S+k}}, \alpha\, \tilde{\beta}_{2^{S+k-1}}\}$ for $k \geq 2$. Therefore,

$$\tilde{\beta}_{2^{S+k}} = \max\left\{\beta_{2^{S+k}}, \alpha\, \beta_{2^{S+k-1}}, \cdots, \alpha^j\, \beta_{2^{S+k-j}}, \cdots, \alpha^k\, \beta_{2^S}\right\}. \tag{4.1}$$

Here the term $\alpha^{k-1}\, \beta_{2^{S+1}}$ can be excluded from the set in braces since it is smaller then the last one. We preserve the previous values: $\tilde{\beta}_{2^s} = \beta_{2^s}$ for $s \leq S$.

Let us show that $\tilde{\beta}_{2^s} \searrow 0$ as $s \to \infty$. For monotonicity we see that $\tilde{\beta}_{2^s} \leq \tilde{\beta}_{2^{s-1}}$ for $s \leq S$ since the sequence $(\beta_n)_{n=0}^{\infty}$ was monotone before the transformation. Also, $\tilde{\beta}_{2^{S+k+1}} = \max\{\beta_{2^{S+k+1}}, \alpha\, \tilde{\beta}_{2^{S+k}}\} \leq \max\{\beta_{2^{S+k}}, \tilde{\beta}_{2^{S+k}}\} = \tilde{\beta}_{2^{S+k}}$ for $k \in \mathbb{N}$. In addition, the general term $\alpha^j\, \beta_{2^{S+k-j}}$ in Eq. 4.1 will be as small as we wish for large enough k. Indeed, let $m = [k/2]$ be the greatest integer at $k/2$. The separate estimation for the cases $0 \leq j \leq m$ and $m + 1 \leq j \leq k$ yields the bound $\tilde{\beta}_{2^{S+k}} \leq \max\left\{\beta_{2^{S+k-m}}, \alpha^{m+1}\beta_{2^S}\right\}$.

Define $\tilde{M}_{2^s} = \exp(2^s \cdot \tilde{\beta}_{2^s})$. Since $2\tilde{\beta}_{2^{s+1}} = 2\max\{\beta_{2^{s+1}}, \alpha\, \tilde{\beta}_{2^s}\} \geq 2\alpha\, \tilde{\beta}_{2^s} > \tilde{\beta}_{2^s}$, as $1/2 < \alpha$, we observe that $\tilde{M}_{2^s} \nearrow$. But we need monotonicity of the whole sequence $(\tilde{M}_n)_{n=1}^{\infty}$. In order to get it, we introduce new intermediate values $\tilde{\beta}_n$ for $2^s < n < 2^{s+1}$ as $\tilde{\beta}_n = \max\{\beta_n, 2^s/n \cdot \tilde{\beta}_{2^s}\}$, whereas the values $\tilde{\beta}_{2^s}$ for $s \in \mathbb{Z}_+$ will not be changed.

If $2^s < n \leq 2^{s+1} - 2$ then $(n+1)\tilde{\beta}_{n+1} = \max\{(n+1)\beta_{n+1}, 2^s\tilde{\beta}_{2^s}\} \geq n\tilde{\beta}_n$, since we had $(n+1)\beta_{n+1} \geq n\beta_n$ for the previous values.

If $n = 2^{s+1} - 1$ then the value $\tilde{\beta}_{n+1}$ is given, so we need to check that $2^{s+1}\tilde{\beta}_{2^{s+1}} \geq (2^{s+1} - 1)\tilde{\beta}_{2^{s+1}-1} = \max\{(2^{s+1} - 1)\beta_{2^{s+1}-1}, 2^s\tilde{\beta}_{2^s}\}$. This is valid due to the monotonicity of $(M_n)_{n=1}^{\infty}$ and $(\tilde{M}_{2^s})_{s=0}^{\infty}$.

We do not require the monotonicity of $\tilde{\beta}_n$. Since at any step we only increase the sequence, we have $\tilde{M}_n \geq m_n$ for all n. Removing the tilde from $\tilde{M}_n$ and $\tilde{\beta}_n$ gives the desired sequence. □

Theorem 4.4 *For every $(M_n)_{n=1}^{\infty}$ of subexponential growth there exists $K(\gamma)$ such that $W_n(K(\gamma)) > M_n$ for all $n \in \mathbb{N}$.*

Proof Let us write γ_k in the form $\gamma_k = \frac{1}{4}\exp(-2^k \cdot a_k)$ for $k \in \mathbb{N}$. If $a_k \geq 0$ and $\sum_{k=1}^{\infty} a_k < \infty$ then the set $K(\gamma)$ is well-defined and is not polar. In addition, as is easy to check, $W_{2^s}(K(\gamma)) = 2\exp\left(2^s \sum_{k=s+1}^{\infty} a_k\right)$.

We use logarithmic subadditivity of Widom's factors. Since $t_{m+r}(K) \leq t_m(K) \cdot t_r(K)$, we have $W_{m+r}(K) \leq W_m(K) \cdot W_r(K)$ for all $m, r \in \mathbb{N}$ and each non-polar compact set K. Let $2^s < n < 2^{s+1}$ for some $s \in \mathbb{N}$. Then n can be represented in the form

$$n = 2^{s+1} - 2^{p_1} - 2^{p_2} - \cdots - 2^{p_m}$$

with $0 \le p_1 < p_2 < \cdots < p_m \le s-1$. Therefore, $W_{2^{s+1}} \le W_n \cdot W_{2^{p_1}} \cdot W_{2^{p_2}} \cdots W_{2^{p_m}} \le W_n \cdot W_1 \cdot W_2 \cdot W_4 \cdots W_{2^{s-1}}$, since $W_k \ge 1$. Here and in the next line we omit the argument $K(\gamma)$ of the Widom factors. In our case, $W_1 \cdot W_2 \cdot W_4 \cdots W_{2^{s-1}} =$

$$2^s \exp\left(\sum_{k=1}^{\infty} a_k + 2\sum_{k=2}^{\infty} a_k + \cdots + 2^{s-1}\sum_{k=s}^{\infty} a_k\right) = 2^s \exp\left(2^s \sum_{k=s}^{\infty} a_k + \sum_{k=1}^{s-1} 2^k a_k - \sum_{k=1}^{\infty} a_k\right).$$

From here, for $2^s < n < 2^{s+1}$ we get

$$W_n(K(\gamma)) \ge 2^{1-s} \exp\left(2^{s+1}\sum_{k=s+2}^{\infty} a_k - 2^s \sum_{k=s}^{\infty} a_k - \sum_{k=1}^{s-1} 2^k a_k + \sum_{k=1}^{\infty} a_k\right). \tag{4.2}$$

We can assume that $(M_n)_{n=1}^{\infty}$ satisfies all conditions given in Lemma 4.3, where $\alpha < 1$ is chosen such that

$$4 - \frac{3}{\alpha(2\alpha - 1)} > \log 2. \tag{4.3}$$

This can be achieved as the expression on the left has the limit 1 as $\alpha \nearrow 1$.

Let us take $a_k = 3(\beta_{2^{k-1}} - \beta_{2^k})$ for $k \in \mathbb{N}$, so $\sum_{k=m}^{\infty} a_k = 3 \cdot \beta_{2^{m-1}}$. Then $W_{2^s}(K(\gamma)) = 2\exp(3 \cdot 2^s \cdot \beta_{2^s})$ which exceeds $M_{2^s} = \exp(2^s \cdot \beta_{2^s})$ for $s \in \mathbb{Z}_+$.

Our next objective is to write the expression in parentheses in Eq. 4.2 in terms of $(\beta_{2^s})_{s=0}^{\infty}$. An easy computation shows that $\sum_{k=1}^{s-1} 2^k a_k = 3\left(-2^{s-1}\beta_{2^{s-1}} + \sum_{k=0}^{s-2} 2^k \beta_{2^k} + \beta_1\right)$ and the whole expression is $3\left(2^{s+1}\beta_{2^{s+1}} - \sum_{k=0}^{s-1} 2^k \beta_{2^k}\right)$. Therefore,

$$W_n(K(\gamma)) \ge \exp\left[3 \cdot 2^{s+1}\beta_{2^{s+1}} - 3\sum_{k=0}^{s-1} 2^k \beta_{2^k} - (s-1)\log 2\right].$$

Since the sequence $(M_n)_{n=1}^{\infty}$ increases and $2^s < n < 2^{s+1}$ for some $s \ge 1$, it suffices to prove that $W_n(K(\gamma)) \ge \exp(2^{s+1}\beta_{2^{s+1}})$ or

$$2^{s+2}\beta_{2^{s+1}} \ge 3\sum_{k=0}^{s-1} 2^k \beta_{2^k} + (s-1)\log 2.$$

By Lemma 4.3, $\beta_{2^k} \le \alpha^{-s-1+k}\beta_{2^{s+1}}$. Therefore,

$$\sum_{k=0}^{s-1} 2^k \beta_{2^k} \le 2^{s+1}\beta_{2^{s+1}} \sum_{k=0}^{s-1} (2\alpha)^{-s-1+k} < 2^{s+1}\beta_{2^{s+1}} \frac{1}{2\alpha(2\alpha-1)}.$$

In this way we reduce the desired inequality to

$$2^{s+1}\beta_{2^{s+1}}\left[2 - \frac{3}{2\alpha(2\alpha-1)}\right] \ge (s-1)\log 2.$$

By Eq. 4.3, the expression in square brackets exceeds $\log 2/2$, so it is enough to check that $2^{s+1}\beta_{2^{s+1}} \ge 2(s-1)$ or $(2\alpha)^{s+1}\beta_1 \ge 2(s-1)$. This is valid since $\beta_1 \ge 1$ and

$\max_{s\ge 1} \frac{2(s-1)}{(2\alpha)^{s+1}} = \frac{1}{4\alpha^4}$ for $\alpha > 3/4$. The condition (4.3) provides that $\alpha > 3/4$ and $4\alpha^4 > 1$. □

In the general case the behavior of $W_n(K(\gamma))$ may be highly irregular.

5 The Irregular Case

Here we combine the previous results. The following example illustrates the construction in the last theorem.

Example 5.1 Suppose we are given an increasing sequence of natural numbers $(s_j)_{j=1}^{\infty}$ and a sequence $(\varepsilon_j)_{j=1}^{\infty}$ of positive numbers with $\varepsilon_1 \le 1$ and $\varepsilon_j \searrow 0$ as $j \to \infty$. Let us take $\gamma_s = \gamma_0 < 1/4$ for $s \ne s_k$ and $\gamma_{s_j} = \gamma_0\, \varepsilon_j$ otherwise. By Eq. 3.1, the set $K(\gamma)$ is not polar if and only if

$$\sum_{j=1}^{\infty} 2^{-s_j} \log \frac{1}{\varepsilon_j} < \infty. \tag{5.1}$$

By Proposition 3.1,

$$W_{2^{s_j}}(K(\gamma)) = 1/2\gamma_0 \cdot \exp\left(2^{s_j} \sum_{k=j+1}^{\infty} 2^{-s_k} \log \frac{1}{\varepsilon_k}\right).$$

If we take, for a given s_j, a large enough value of s_{j+1},then $W_{2^{s_j}}(K(\gamma))$ can be obtained as closed to $1/2\gamma_0$ as we wish.

On the other hand,

$$W_{2^{s_j-1}}(K(\gamma)) = \frac{1}{2} \exp\left(2^{s_j-1} \sum_{n=s_j}^{\infty} 2^{-n} \log \frac{1}{\gamma_n}\right).$$

Taking into account only the first term in the series, we get

$$W_{2^{s_j-1}}(K(\gamma)) > \frac{1}{2} \frac{1}{\sqrt{\gamma_0\, \varepsilon_j}} > \frac{1}{\sqrt{\varepsilon_j}},$$

which may be large for small ε_j satisfying (5.1).

Let us construct a set $K(\gamma)$ for which both behaviours of subsequences (as in Theorem 3.4 and Theorem 4.4) are possible.

Theorem 5.2 *For any sequences $(\sigma_j)_{j=0}^{\infty}$ with $\sigma_j \searrow 0$ and $(M_n)_{n=1}^{\infty}$ of subexponential growth with $M_n \to \infty$ there exists a sequence $(\gamma_s)_{s=1}^{\infty}$ such that for the corresponding set $K(\gamma)$ there are two sequences $(s_j)_{j=1}^{\infty}$ and $(q_j)_{j=1}^{\infty}$ with $W_{2^{s_j}}(K(\gamma)) < 2(1+\sigma_j)$ and $W_{2^{q_j}}(K(\gamma)) > M_{2^{q_j}}$ for all $j \in \mathbb{N}$.*

Proof Without loss of generality we can assume $\sigma_1 \le 1$ and $M_n \ge 1$ for all n. For the sequences $(s_j)_{j=1}^{\infty}$, $(\varepsilon_j)_{j=1}^{\infty}$ that will be specified later, we define $\gamma_s = (4\sqrt{1+\sigma_j})^{-1}$ for $s_j < s < s_{j+1}$ and $\gamma_{s_j} = \varepsilon_j\,(4\sqrt{1+\sigma_j})^{-1}$. Also we take $q_j = s_j - 1$. Then, as above,

$$W_{2^{q_j}}(K(\gamma)) > \frac{1}{2} \frac{1}{\sqrt{\gamma_{s_j}}} > \frac{1}{\sqrt{\varepsilon_j}},$$

so we can take $\varepsilon_j = M_{2^{s_j-1}}^{-2}$.

On the other hand, $W_{2^{s_j}}(K(\gamma)) = \frac{1}{2}\exp\left[2^{s_j}\sum_{n=s_j+1}^{\infty} 2^{-n}\log 1/\gamma_n\right]$ with

$$\sum_{n=s_j+1}^{\infty} 2^{-n}\log\frac{1}{\gamma_n} = \sum_{n=s_j+1, n\neq s_k}^{\infty} 2^{-n}\log\frac{1}{\gamma_n} + \sum_{k=j+1}^{\infty} 2^{-s_k}\log(4\sqrt{1+\sigma_k}) + \sum_{k=j+1}^{\infty} 2^{-s_k}\log\frac{1}{\varepsilon_k}.$$

We combine the first two sums on the right:

$$\sum_{k=j+1}^{\infty}\sum_{n=s_k+1}^{s_{k+1}} 2^{-n}\log(4\sqrt{1+\sigma_k}) < 2^{-s_j}\log(4\sqrt{1+\sigma_j}),$$

since $(\sigma_k)_{k=0}^{\infty}$ decreases. From here,

$$W_{2^{s_j}}(K(\gamma)) < 2\sqrt{1+\sigma_j}\,\exp\left[2^{s_j}\sum_{k=j+1}^{\infty} 2^{-s_k}\log\frac{1}{\varepsilon_k}\right]$$

and we have the desired result if the expression in square brackets does not exceed $\log(\sqrt{1+\sigma_j})$ or, by definition of ε_k,

$$\sum_{k=j+1}^{\infty} 2^{s_j-s_k}\log M_{2^{s_k-1}} < \frac{1}{4}\log(1+\sigma_j). \tag{5.2}$$

This can be achieved if we ensure for all k

$$2^{s_{k-1}-s_k}\log M_{2^{s_k-1}} < \frac{1}{8}\log(1+\sigma_k). \tag{5.3}$$

Indeed, provided (5.3), the $k-$th term in the series above is

$$2^{s_j-s_{k-1}}2^{s_{k-1}-s_k}\log M_{2^{s_k-1}} < 2^{s_j-s_{k-1}}\frac{1}{8}\log(1+\sigma_k) < 2^{s_j-s_{k-1}}\frac{1}{8}\log(1+\sigma_j),$$

by monotonicity of $(\sigma_k)_{k=0}^{\infty}$. Summing these terms, we get (5.2).

Thus it remains to choose $(s_k)_{k=1}^{\infty}$ satisfying (5.3). This can be done recursively since $(M_n)_{n=1}^{\infty}$ has subexponential growth and $2^{-s_k+1}\log M_{2^{s_k-1}}$ can be taken smaller than $2^{-s_{k-1}-2}\log(1+\sigma_k)$ for large enough s_k. Clearly, (5.2) implies (5.1). Hence the set $K(\gamma)$ is well-defined and is not polar. □

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